

Nonlinear Systems for Highly Compressive Elasticity Simulations

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Abstract

The standard equations of linear elasticity have served well over many generations to describe small displacements of elastic systems subject to a variety of forces and/or stresses. However, computations quickly reveal the limitations of these equations in modelling even moderately large displacements. Very general nonlinear approaches to finite elasticity exist in the literature but are often somewhat inaccessible for use in practical applications. In the present paper we undertake a preliminary discussion of a specific, simple class of nonlinear systems which appear to be well adapted to computational implementation in compressive elasticity simulations. Both analytical studies and computational simulations are presented.

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1 Background in Linear Elasticity

The subject of linear elasticity possesses a vast and profound literature; we do not pretend any sort of complete coverage here. The reader is referred to [15], [5], [11] for just a small sampling of what is available. In what follows we present our own quick summary of the subject.

We consider an isotropic linear elastic solid in R^m , where $m = 2$ or 3 . The undeformed body occupies a region R_0 in R^m with piecewise smooth boundary B_0 . We denote points in R^m by capital latin letters, X, Y , etc. The unit exterior normal at a point $X \in B_0$ will be denoted by $N = N(X)$. In deformation the region R_0 is carried into a new region R by means of a *deformation map* X sufficiently smooth, at least, so that it lies in $H^1(R_0)$. We write

$$X(X) \equiv X + \Xi(X), \quad X \in R_0,$$

where $\Xi(X)$, the *displacement field*, has components ξ, η , or ξ, η, ζ , according as $m = 2$ or 3 . The Jacobian matrix of $X(X)$ is then $\mathbf{I} + \nabla\Xi(X)$, \mathbf{I} denoting the $m \times m$ identity matrix. Physical realism would ordinarily require $X(X)$ to be bijective with $\det(\mathbf{I} + \nabla\Xi(X))$ positive throughout R_0 . Some studies in *finite elasticity* (cf. [6], e.g.) have allowed for cases of cavitation and material interpenetration, however.

Linear elasticity is primarily concerned with the symmetric part of $\nabla\Xi(X)$, the *strain matrix*

$$\mathbf{E}(X) = \frac{1}{2}(\nabla\Xi(X) + \nabla\Xi(X)^*).$$

Distributed (body) *forces* may be represented as square integrable vector functions $F(X)$ defined for $X \in R_0$. For a given displacement $\Xi(X)$ the corresponding *work integral* is

$$\int_{R_0} F(X)^* \Xi(X) dX,$$

where dX represents the standard area or volume measure for $m = 2$ or 3 , respectively. If we assume $\Xi(X) \in H^1(R_0)$ then (cf. [10]) it has a boundary trace in $L^2(B_0) = L^2(B_0)$ and we can consider square integrable *boundary forces* (tractions) $G(X)$ with support in B_0 for which the corresponding work integral is

$$\int_{B_0} G(X)^* \Xi(X) ds,$$

wherein ds represents arc length or area measure on \mathbf{B}_0 according as $m = 2$ or 3 , respectively. The forces $F(X)$, $G(X)$ are *statically admissible* if they satisfy the conditions

$$\int_{R_0} F(X) dX + \int_{B_0} G(X) ds = 0, \quad \int_{R_0} F(X) \times X dX + \int_{B_0} G(X) \times X ds = 0.$$

One, or both, of these conditions may be omitted if there are side conditions, which may be boundary conditions, requiring that one or more points in R_0 must have fixed values for all displacements $\Xi(X)$ under consideration.

Linear elasticity posits a linear *constitutive law* whereby each displacement function $\Xi(X)$ engenders a corresponding reactive stress via a linear relationship

$$\mathbf{S}(X) = -\Sigma(\nabla\Xi(X)), \quad (1.1)$$

Σ denoting a symmetric linear operator on $m \times m$ matrices \mathbf{M} with the properties:

- i) $\Sigma(\mathbf{M})$ is symmetric for any $m \times m$ matrix \mathbf{M} ;
- ii) $\Sigma(\mathbf{M}^*) = \Sigma(\mathbf{M})^*$ (\mathbf{M} antisymmetric $\Rightarrow \Sigma(\mathbf{M}) = 0$);
- iii) $\text{Tr } \Sigma(\mathbf{M}_1) \mathbf{M}_2 = \text{Tr } \Sigma(\mathbf{M}_2) \mathbf{M}_1$ (operator symmetry);
- iv) $\mathbf{M} \neq 0$ symmetric $\Rightarrow \text{Tr } \Sigma(\mathbf{M}) \mathbf{M} > 0$ (positivity).

Different types of materials have different associated $\Sigma(\mathbf{M})$; in all cases the *elastic potential energy* for a displacement $\Xi(X)$ is given by the integral

$$E_{\Sigma}(\Xi) = \frac{1}{2} \int_{R_0} \text{Tr } \Sigma(\mathbf{E}(X)) \mathbf{E}(X) dX = \frac{1}{2} \int_{R_0} \text{Tr } \Sigma(\nabla \Xi(X)) \nabla \Xi(X) dX. \quad (1.2)$$

Linear isotropic materials correspond to

$$\Sigma(\mathbf{E}(X)) = 2\mu \mathbf{E}(X) + \lambda \mathbf{D}(X),$$

where μ and λ are the Lamé constants and $\mathbf{D}(X) = (\text{div} \mathbf{E}(X)) \mathbf{I} = (\text{div} \Xi(X)) \mathbf{I}$. This is the case studied here, for which (1.2) becomes

$$E(\Xi) = \frac{1}{2} \int_{R_0} \text{Tr} [(2\mu \mathbf{E}(X) + \lambda \mathbf{D}(X)) \mathbf{E}(X)] dX. \quad (1.3)$$

Given statically admissible square integrable forces $F(X)$, $G(X)$ on R_0 , B_0 , respectively, it is well known that the resulting displacement $\Xi(X)$, unique modulo translation and infinitesimal rotation, is obtained from minimization of the Lagrangian (cf. (1.2))

$$H_0(\Xi) = E(\Xi) - \int_{R_0} F(X)^* \Xi(X) dX - \int_{B_0} G(X)^* \Xi(X) ds. \quad (1.4)$$

As detailed elsewhere (see, e.g. [12], [13]) such minimization leads to the system of partial differential equations

$$\text{div}(\Sigma(\mathbf{E}(X))) + F(X) = 0, \quad X \in R_0, \quad (1.5)$$

along with (free) boundary conditions

$$\Sigma(\mathbf{E}(X)) N(X) - G(X) = 0, \quad X \in B_0. \quad (1.6)$$

2 Some Shortcomings of the Linear Model

It is well known that the equations (1.5), (1.6) are accurate only for very small displacements $\Xi(X)$. The inaccuracy of the linear model for large displacements is not only *quantitative* but entails major *qualitative* errors as well.

The computational examples which we use to illustrate this all deal with cases where $m = 2$ and, in (1.5) and (1.6), $F(X) \equiv 0$ while, for constant $p > 0$,

$$G(X) \equiv \begin{cases} -p N(X), & X \in B_1 \subset B_0, \\ 0, & X \in B_0 - B_1, \end{cases} \quad (2.1)$$

represents a uniform external pressure applied to R_0 , directed toward the interior of that set, with support in the closed measurable subset B_1 of B_0 . In this case the boundary condition (1.6) includes

$$\Sigma(\mathbf{E}(X)) N(X) + p N(X) = 0, \quad X \in B_1, \quad (2.2)$$

so that, for $X \in B_1$, $N(X)$ is an eigenvector of $\Sigma(\mathbf{E}(X))$ with eigenvalue $-p$. Specifically we consider the case wherein $R_0 \subset R^2$ is the subregion of the square $|x| \leq 1$, $|y| \leq 1$, corresponding to $x^2 + y^2 \geq r_0^2$, $0 < r_0 < 1$. The circle $x^2 + y^2 = r_0^2$ serves as B_1 while $B_0 - B_1$ is given by $|x| = 1$ or $|y| = 1$. We will assume the latter is a *fixed* boundary, i.e.,

$$\Xi(X) = 0, \quad X \in B_0 - B_1, \quad (2.3)$$

while (2.2) holds on B_1 , i.e., on the circle $x^2 + y^2 = r_0^2$. The relevant system of partial differential equations is, of course, (1.5) with $F(X) \equiv 0$, $X \in R_0$.

Our computations were carried out using linear finite elements on triangles, the totality of which constitute a triangular decomposition of a polygonally bounded region \tilde{R}_0 approximating R_0 . For background on this and related methods the reader is referred, e.g., to [7], [4]. To minimize computation time and to facilitate inspection of the resulting plots, we took advantage of obvious symmetries and carried out the computations with triangulations of just the intersection of the indicated region R_0 with the first quadrant in R^2 . Our graphics are also restricted to this region; we rely on the reader's imagination to extend these plots to all four quadrants.

In Figure 1 we show the original, undisplaced elastic body, equivalently the region R_0 , via the triangulated approximation used in the finite element computations.

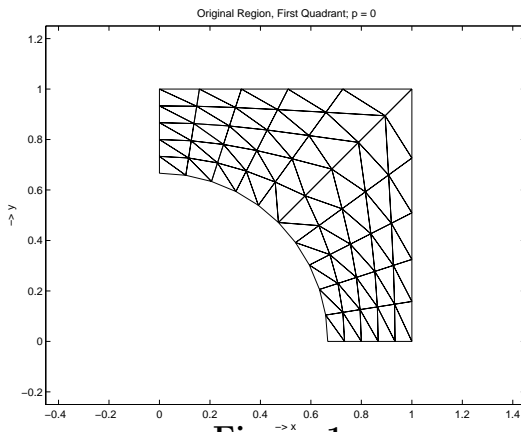
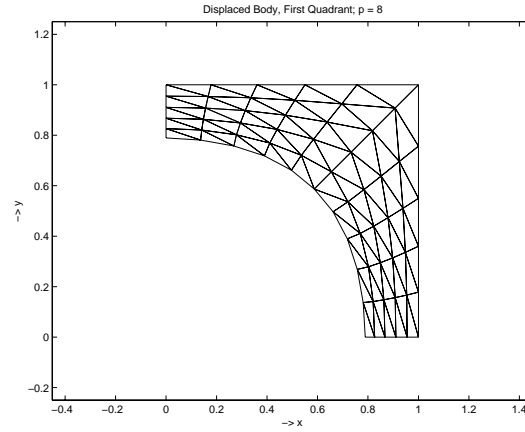
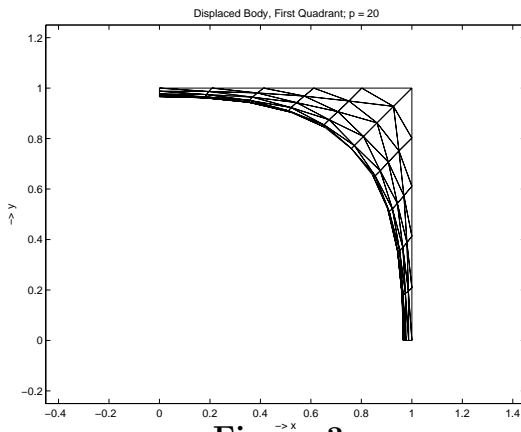
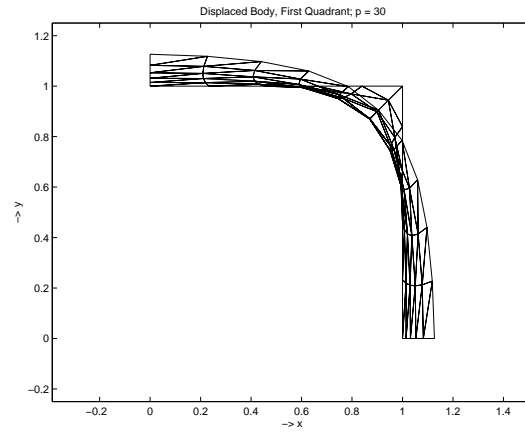
**Figure 1****Figure 2**

Figure 2 shows the first quadrant portion of the same elastic body, now subject to a moderate pressure acting on the surface of the central vacuole. This is clearly the type of application for which linear elasticity is appropriate, at least in the qualitative sense. We again remind the reader that the outer boundary, $|x| = 1$ or $|y| = 1$ is fixed in these computations so that the pressure in the interior vacuole should tend to press the elastic material against this boundary.

**Figure 3****Figure 4**

In Figures 3 and 4 we continue to show the same elastic body but with progressively larger pressures applied in the interior vacuole. These plots are increasingly implausible from the physical point of view. Figure 3 shows the elastic body compressed to almost zero thickness at the top and at the right while the portion of the body toward the upper right corner is relatively unstressed. With an even larger internal pressure we come to the utterly non-physical situation shown in Figure 4 where the material has been forced to interpenetrate itself; in this case $\det(\mathbf{I} + \nabla \Xi)$ does not maintain positivity.

The reason for this unphysical behavior of the linear elastic body is not hard to find. Since the model is linear and the only forcing term is the pressure p

applied at the inner surface, the displacement, at any point X , must be a linear function of p . This means that the displacements shown in Figure 3 are $2\frac{1}{2}$ times those in Figure 2 and those in Figure 4 are $3\frac{3}{4}$ those in Figure 2, any other features, such as the fixed outer boundary, notwithstanding. The model has no means to recognize the influence which the fixed outer boundary should increasingly have, with increasing p , on the displacements of the system.

3 Compression Sensitive Nonlinearities

The foregoing observations lead us into the field of nonlinear, in particular *compressible*, elasticity and its extensive literature; see, e.g., [1] [2], [8], [3], [6], [9], [11], to cite just a few most relevant here. Since compression is signalled by small values of $\det(\mathbf{I} + \nabla\Xi(X))$, we are prompted to consider modifications to the Lagrangian which involve this quantity. To this end we begin with the Lagrangian for the linear elasticity model with small pressures

$$\frac{1}{2} \int_{R_0} \text{Tr}(\Sigma(\mathbf{E}(X))\mathbf{E}(X)) dX - p \int_{B_0} \Xi(X) ds. \quad (3.1)$$

So formulated, the Lagrangian ignores the change from B_0 to B_p resulting from the applied pressure. For large positive p this can be quite significant because the area/length of B_p may be substantially larger than that of B_0 . Alternatively we can use

$$\frac{1}{2} \int_{R_0} \text{Tr}(\Sigma(\mathbf{E}(X))\mathbf{E}(X)) dX - \frac{p}{2} \text{meas}(V_p), \quad (3.2)$$

where “meas” denotes the volume/area of the pressurized vacuole, $V_p = X(V)$.

The essential nonlinear modification studied here modifies (3.2) to

$$\frac{1}{2} \int_{R_0} [\text{Tr}(\Sigma(\mathbf{E}(X))\mathbf{E}(X)) + \Phi(\det(\mathbf{I} + \nabla\Xi(X)))] dX - \frac{p}{2} \text{area}(X(V)) \quad (3.3)$$

with the function Φ chosen so that $\Phi(\det(\mathbf{I} + \nabla\Xi(X)))$ grows to infinity as its argument $\det(\mathbf{I} + \nabla\Xi(X))$ becomes small. In this we largely follow the description of compressible elasticity used, e.g., in [6]. Experience indicates that we should stress the following point: *our purpose in this paper is not to advocate the model based on (3.2) for general use in nonlinear elasticity studies*. Rather, for certain functions Φ we want to explore the consequences of computational use of such models and we want to examine some of the mathematical properties such models possess.

There are many possibilities for $\Phi(d)$, e.g.: $-\log(d)$, d^{-k} , $k = 1, 2, \dots$, etc.. This list is by no means exhaustive. A variety of modifications suggest themselves as desirable in various circumstances. In our test problem the choice

$\Phi(r) = -\log(r)$ suffers from the disadvantage that, since its derivative, $-1/r$, is not zero at $r = 1$, which corresponds to $p = 0$, it modifies the equilibrium configuration all by itself. A better choice is $\Phi(r) = r - 1 - \log(r)$ which is positive for all $r \neq 1$ and has minimum value, and derivative, both equal to zero at $r = 1$. In the simulations of this section the Lagrangian (3.3) was minimized for various values of p and $\Phi(d)$ of the form $\Phi(d) = \alpha(d - 2 + d^{-1})$, $\alpha > 0$. Some results are shown in Figures 9 through 12. Figures 9 and 10 show the results for moderate and high pressure, respectively, while Figures 11 and 12 correspond to application of very high pressures. The effects are very clear; even under very high pressures the compressed elastic body does not undergo interpenetration and remains within the physically plausible region. These results may be compared with analogous results using the Blatz–Ko foam rubber model [8] reported in [3]. Our version of $\Phi(d)$ corresponds, in respect to *polyconvexity*, to the assumptions made in [2] but not with respect to the growth assumptions made in that paper to assure existence of minimizers.

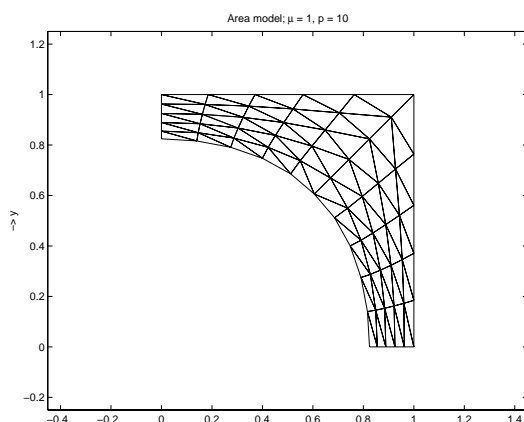


Figure 9

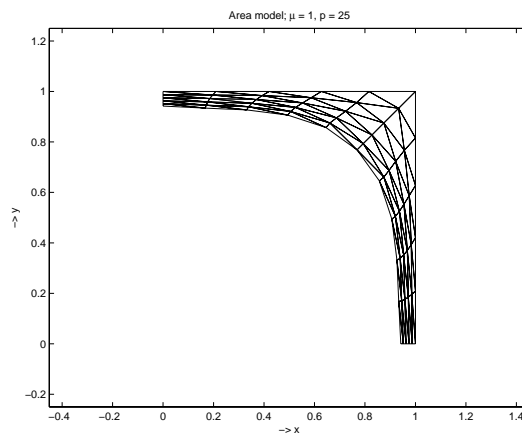


Figure 10

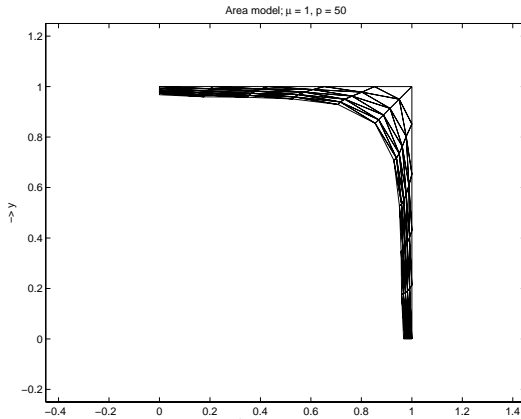


Figure 11

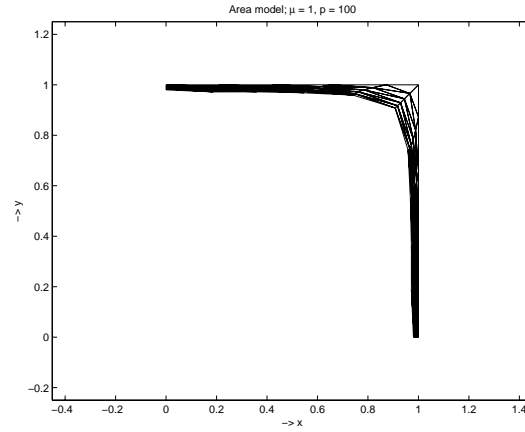


Figure 12

Much remains to be done toward validating the use of Lagrangians of the form (3.3) as an appropriate methodology for studying “soft” elastic materials under high compressive loads. The next section offers a step in that direction.

4 A Rotationally Invariant Problem for $m = 2$

For an analytical study we consider the special case of an elastic body in R^2 occupying, in unforced equilibrium, an annular region described in polar coordinates r, θ , by $r_0 \leq r \leq 1$ for some r_0 , $0 < r_0 < 1$. The central vacuole $0 \leq r < r_0$ is assumed pressurized to a level p . The problem differs from that studied numerically in the preceding section only to the extent that the fixed outer boundary is now the circle $r = 1$. Thus we have the boundary condition

$$\rho(1) = 0. \quad (4.1)$$

For displacements $\rho(r)$ independent of θ , and obeying (4.1), conversion to polar coordinates and division by 2π changes the Lagrangian (3.3) to

$$\int_{r_0}^1 \left[\frac{(\lambda + \mu)}{2} \left(\rho'(r) + \frac{\rho(r)}{r} \right)^2 + \frac{\mu}{2} \left(\rho'(r) - \frac{\rho(r)}{r} \right)^2 + \Phi(r) \right] r dr - p (r_0 + \rho(r_0)). \quad (4.2)$$

(We could also replace the last term by $p(r + r_0)^2/2$ as in our earlier discussion; this changes little in the discussion to follow.)

The term $\det(\mathbf{I} + \nabla \Xi(X))$ now becomes

$$d(r, \rho(r), \rho'(r)) \equiv \left(1 + \frac{\rho(r)}{r} \right) (1 + \rho'(r)) = \frac{1}{2r} \frac{d}{dr} (r + \rho(r))^2. \quad (4.3)$$

Following our discussion at the end of the preceding section, we work with

$$\Phi(d) = \alpha \left(\phi(d) - \left(\phi(1) + \phi'(1)(d - 1) \right) \right), \quad (4.4)$$

which implies $\Phi(1) = \Phi'(1) = 0$. We initially make

Assumption 4.1 *The function $\phi(d)$ is twice continuously differentiable with*

$$\phi''(d) > 0, \quad (d^2 \phi''(d))' \leq 0, \quad d > 0; \quad \lim_{d \downarrow 0} \phi(d) = \infty. \quad (4.5)$$

Denoting a small variation in $\rho(r)$ by $\delta\rho(r)$ and setting the variation of (4.2) equal to zero yields

$$\begin{aligned} \int_{r_0}^1 & \left[(\lambda + \mu) \left(\rho'(r) + \frac{\rho(r)}{r} \right) \left(\delta\rho'(r) + \frac{\delta\rho(r)}{r} \right) + \mu \left(\rho'(r) - \frac{\rho(r)}{r} \right) \left(\delta\rho'(r) - \frac{\delta\rho(r)}{r} \right) \right. \\ & \left. + \alpha \left(\frac{\partial\Phi(d)}{\partial\rho} \delta\rho(r) + \frac{\partial\Phi(d)}{\partial\rho'} \delta\rho'(r) \right) \right] r dr - p \delta\rho(r_0) = 0. \end{aligned}$$

Separating terms involving $\delta\rho(r)$ and $\delta\rho'(r)$ we have

$$\begin{aligned} & \int_{r_0}^1 \left[\lambda \rho'(r) + (\lambda + 2\mu) \frac{\rho(r)}{r} + \alpha r \frac{\partial\Phi(d)}{\partial\rho} \right] \delta\rho(r) dr - p \delta\rho(r_0) \\ & + \int_{r_0}^1 \left[(\lambda + 2\mu) r \rho'(r) + \lambda \rho(r) + \alpha r \frac{\partial\Phi(d)}{\partial\rho'} \right] \delta\rho'(r) dr = 0. \end{aligned} \quad (4.6)$$

Applying integration by parts to the second integral and using (4.1) we have

$$\begin{aligned} & \int_{r_0}^1 \left[(\lambda + 2\mu) \left(\frac{\rho(r)}{r} - \frac{d}{dr} (r \rho'(r)) \right) + \alpha r \frac{\partial\Phi(d)}{\partial\rho} - \alpha \frac{d}{dr} \left(r \frac{\partial\Phi(d)}{\partial\rho'} \right) \right] \delta\rho(r) dr \\ & - \left[(\lambda + 2\mu) r_0 \rho'(r_0) + \lambda \rho(r_0) + \alpha r_0 \frac{\partial\Phi(d)}{\partial\rho'} (r_0, \rho(r_0), \rho'(r_0)) + p \right] \delta\rho(r_0) = 0. \end{aligned}$$

From this, by the usual arguments, we have the differential equation

$$(\lambda + 2\mu) \left(\frac{d}{dr} (r \rho'(r)) - \frac{\rho(r)}{r} \right) - \alpha r \frac{\partial\Phi(d)}{\partial\rho} + \alpha \frac{d}{dr} \left(r \frac{\partial\Phi(d)}{\partial\rho'} \right) = 0 \quad (4.7)$$

and the boundary condition, supplementing (4.1),

$$(\lambda + 2\mu) r_0 \rho'(r_0) + \lambda \rho(r_0) + \alpha r_0 \frac{\partial\Phi(d)}{\partial\rho'} (r_0, \rho(r_0), \rho'(r_0)) + p = 0. \quad (4.8)$$

Now let us note that

$$r \frac{\partial\Phi(d)}{\partial\rho} = \Phi'(d) (1 + \rho'), \quad r \frac{\partial\Phi(d)}{\partial\rho'} = \Phi'(d) (r + \rho). \quad (4.9)$$

Thus

$$\alpha r \frac{\partial \Phi(d)}{\partial \rho} - \alpha \frac{d}{dr} \left(r \frac{\partial \Phi(d)}{\partial \rho'} \right) = \alpha \Phi'(d)(1+\rho') - \alpha \frac{d}{dr} \left(\Phi'(d)(r+\rho) \right). \quad (4.10)$$

In the process of arriving at (4.10) it is clear that the term $\phi(1)+\phi'(1)(d-1)$ in (4.4), whose derivative with respect to d is the constant $\phi'(1)$, is reduced to zero. Thus (4.7) can be replaced by

$$(\lambda + 2\mu) \left(\frac{d}{dr} (r\rho'(r)) - \frac{\rho(r)}{r} \right) - \alpha r \frac{\partial \phi(d)}{\partial \rho} + \alpha \frac{d}{dr} \left(r \frac{\partial \phi(d)}{\partial \rho'} \right) = 0. \quad (4.11)$$

Now, suppressing α for the moment and treating ϕ as we did for Φ in (4.9), we compute

$$\begin{aligned} -r \frac{\partial \phi(d)}{\partial \rho} + \frac{d}{dr} \left(r \frac{\partial \phi(d)}{\partial \rho'} \right) &= -\phi'(d) \left(1 + \rho' \right) + \frac{d}{dr} \left(r \phi'(d) \left(1 + \frac{\rho}{r} \right) \right) \\ &= -\phi''(d)(1+\rho') + \left(\phi'(d) \left(1 + \frac{\rho}{r} \right) + r \frac{d\phi'(d)}{dr} \left(1 + \frac{\rho}{r} \right) + \phi'(d) \left(\rho' - \frac{\rho}{r} \right) \right) \\ &= r \frac{d\phi'(d)}{dr} \left(1 + \frac{\rho}{r} \right) = r \phi''(d) \frac{d}{dr} \left(\left(1 + \frac{\rho}{r} \right) (1 + \rho') \right) \left(1 + \frac{\rho}{r} \right) \\ &= d \phi''(d) \left(\rho' - \frac{\rho}{r} \right) + r \phi''(d) \left(1 + \frac{\rho}{r} \right)^2 \rho''. \end{aligned}$$

Then (4.11) becomes

$$(\lambda + 2\mu) \left(\frac{d}{dr} (r\rho'(r)) - \frac{\rho(r)}{r} \right) + \alpha \left(d \phi''(d) \left(\rho' - \frac{\rho}{r} \right) + r \phi''(d) \frac{d^2 \rho''}{(1 + \rho')^2} \right) = 0. \quad (4.12)$$

Let us set $\sigma(r) = r + \rho(r)$. Multiplying by $\frac{(d\sigma/dr)^2}{(d^2\phi''(d))}$ and rearranging, using $\sigma''(r) \equiv \rho''(r)$, $\sigma' - \sigma/r = \rho' - \rho/r$ and the definition (4.3) of d , we have

$$\begin{aligned} 0 &= \alpha r \frac{d^2 \sigma}{dr^2} + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} \left(r \frac{d^2 \sigma}{dr^2} + \frac{d\sigma}{dr} - \frac{\sigma}{r} \right) + \frac{\alpha r}{\sigma} \frac{d\sigma}{dr} \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right) = \\ &\alpha r \frac{d^2 \sigma}{dr^2} + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} \left(r \frac{d^2 \sigma}{dr^2} + \frac{d\sigma}{dr} - \frac{\sigma}{r} \right) + \frac{\alpha r}{\sigma} \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right)^2 + \alpha \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right) \end{aligned} \quad (4.13)$$

from which, using the identity

$$r \frac{d^2 \sigma}{dr^2} = r \frac{d}{dr} \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right) + \frac{d\sigma}{dr} - \frac{\sigma}{r}, \quad (4.14)$$

we obtain

$$\frac{d}{dr} \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right) + \frac{1}{r} \left(\frac{2\alpha + 2(\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} + \frac{\alpha r}{\sigma} \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right)}{\alpha + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))}} \right) \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right) = 0.$$

This, with

$$\tau = \frac{d\sigma}{dr} - \frac{\sigma}{r}, \quad (4.15)$$

can be rewritten as the first order system

$$-\frac{d\tau}{dr} = \frac{2}{r} \tau + \left(\frac{\alpha}{\sigma \left(\alpha + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} \right)} \right) \tau^2, \quad \frac{d\sigma}{dr} = \left(\frac{\sigma}{r} + \tau \right). \quad (4.16)$$

Integrating the second equation of (4.16) with $\sigma(1) = 1$ we have

$$\sigma(r) = r \left(1 - \int_r^1 \frac{\tau(s)}{s} ds \right). \quad (4.17)$$

The term $d^2\phi''(d)$ plays a significant role in (4.16). For the “canonical” cases $\phi(d) = -\log(d)$, $\phi(d) = d^{-k}$, $k = 1, 2, 3, \dots$, we have, respectively, $d^2\phi''(d) = 1$, $k(k+1)d^{-k}$, $k = 1, 2, 3, \dots$. Clearly $\phi(d) = -\log(d)$ is particularly advantageous with respect to simplicity of the first equation in (4.16).

Since it is difficult to obtain specific information in studying (4.16), (4.8) and (4.1) as a two point boundary value problem, we will, instead, study solutions of the system (4.16), in the context of a “terminal value problem” with boundary conditions $\sigma(1) = 1$, $\tau(1) = -v$, $v \in [0, 1]$, $(\sigma'(1) = 1 - v)$, ultimately relating its solutions to solutions of the two point boundary value problem.

Theorem 4.1 *Let $\sigma(1) = 1$ and let a terminal value $\tau(1) = -v$, $v \in [0, 1]$ equivalently $\sigma(1) = 1 - v$, be given. Then the solutions $\tau(r)$, $\sigma(r)$ of (4.16) persist over the interval $[r_0, 1]$ with $\tau(r)$ remaining negative, $\sigma(r)$ remaining positive on that interval and $\rho(r) > 0$ on $[r_0, 1]$.*

Proof From the boundary conditions at $r = 1$ there is a minimal \hat{r} , $r_0 \leq \hat{r} < 1$, such that $\tau(r)$, $\sigma(r)$ exist on $(\hat{r}, 1]$ and $\tau(r) < 0$ on that interval. Since $\sigma(r) = r + \rho(r)$ (4.17) gives

$$\rho(r) = -r \int_r^1 \frac{\tau(s)}{s} ds \quad (4.18)$$

and, from the negativity of $\tau(r)$ on $(\hat{r}, 1]$ we conclude $\rho(r) > 0$, $1 \geq \sigma(r) > r > 0$, over the interval $(\hat{r}, 1)$.

Using the positivity of $\phi''(d)$ from Assumption 4.1, on the interval $(\hat{r}, 1]$ the coefficient of τ^2 in (4.16) lies in the interval $(0, 1/r)$ and therefore, on that interval,

$$-r \frac{d\tau}{dr} \leq 2\tau + \tau^2. \quad (4.19)$$

It follows that $\tau(r)$ lies below the solution curve $2c/(r^2 - c)$ of the differential equation obtained from (4.19) by replacing \leq by $=$, c being determined from $\tau(1)$ by $c = \frac{\tau(1)}{2+\tau(1)} < 0$. Thus we have the inequality

$$\tau(r) \leq \frac{2c}{r^2 - c} = \frac{2\left(\frac{\tau(1)}{2+\tau(1)}\right)}{r^2 - \frac{\tau(1)}{2+\tau(1)}} = \frac{2\tau(1)}{2r^2 - (1-r^2)\tau(1)}, \quad \hat{r} \leq r \leq 1.$$

On the other hand we certainly have

$$-r \frac{d\tau}{dr} \geq 2\tau \Rightarrow \tau(r) \geq \frac{\tau(1)}{r^2}, \quad \hat{r} \leq r \leq 1.$$

Since these estimates, which guarantee $\tau(r) < 0$, are valid as long as $\tau(r) < 0$, we conclude by a standard continuation argument that the interval $(\hat{r}, 1]$ extends, first of all to $[\hat{r}, 1]$, and then to the whole interval $[r_0, 1]$ and $\tau(r) = \frac{d\sigma(r)}{dr} - \frac{\sigma(r)}{r}$ remains well defined and negative over that interval. Then we conclude, as before, that $\sigma(r)$ is well defined and $\geq r$ on $[r_0, 1]$, > 1 on $[r_0, 1)$, while $\rho(r)$ is well defined and ≥ 0 on $[r_0, 1]$, > 0 on $[r_0, 1)$. Further we note that these solutions extend continuously to the case $\tau(r) \equiv 0$, $\sigma(r) \equiv r$, $\rho(r) \equiv 0$, corresponding to $\tau(1) = v = 0$, $\sigma(1) = 1$. This completes the proof.

Application of standard arguments to (4.17) and (4.16) enable us to see, for $r_0 \leq r \leq 1$, that $\tau(r)$, $\tau'(r)$, $\sigma(r)$, $\sigma'(r)$, $\rho(r)$, $\rho'(r)$ are continuous functions of $v = \tau(1)$ in the range indicated.

Theorem 4.2 *Let $\alpha > 0$. Then solutions $\sigma(r)$, as discussed in Theorem 4.1, with $\sigma(1) = 1$, $0 < \sigma'(1) \leq 1$ have the property*

$$\sigma'(r) = 1 + \rho'(r) > 0, \quad r \in [r_0, 1]. \quad (4.20)$$

As a consequence the determinant (4.3) remains positive on this interval.

Proof The first line of (4.13) can be rearranged to show that

$$r \left(\alpha + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} \right) \frac{d^2\sigma}{dr^2} + \left((\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} + \frac{\alpha r d\sigma}{\sigma dr} \right) \left(\frac{d\sigma}{dr} - \frac{\sigma}{r} \right)$$

vanishes. Dividing by $r \left(\alpha + (\lambda + 2\mu) \frac{(d\sigma/dr)^2}{(d^2\phi''(d))} \right)$ the equation takes the form

$$\frac{d\sigma'(r)}{dr} + \psi(r, \sigma(r), \sigma'(r)) \sigma'(r) = 0. \quad (4.21)$$

We emphasize that we do not need to demonstrate the existence of a solution of this equation; our analysis of the system (4.16) shows that $\sigma(r)$ and $\tau(r)$, hence also $\sigma'(r)$, are continuous, hence bounded, as functions of $r \in [r_0, 1]$ for $\sigma(1) = 1$ and all values of $\tau(1)$. From (4.21) and the form of ψ it follows that $\psi(r, \sigma(r), \sigma'(r))$ is also continuous and bounded on that interval. Letting $\Psi(r) = \Psi(r, \sigma(r), \sigma'(r))$ be an r -antiderivative of that function, we have $\sigma'(r) = c e^{-\Psi(r)}$ for some constant c and thus

$$\sigma'(r) = e^{\left(\Psi(1) - \Psi(r)\right)} \sigma'(1), \quad r \in [r_0, 1].$$

This clearly implies (4.20). The positivity of the determinant, completing the proof of the theorem, then follows from the already demonstrated positivity of $\sigma(r)$ and the fact that (4.3) is equivalent to

$$d(r, \rho(r), \rho'(r)) = \frac{\sigma(r) \sigma'(r)}{r}.$$

Let us denote the triangular region in the r, σ plane bounded by $r = r_0$, $\sigma = 1$ and $\sigma = r$ as Δ_σ . Further, we denote the region in the r, ρ plane bounded by $r = r_0$, the line $\rho = 0$ and the line $\rho = 1 - r$ by Δ_ρ and the region in the r, τ plane bounded by $r = r_0$, $\tau = 0$ and $\tau = -1/r$ by Δ_τ . Our next objective is to show that these regions are simply covered by the curves $(r, \sigma(r))$, $(r, \rho(r))$ and $(r, \tau(r))$, respectively.

Theorem 4.3 *As $\sigma'(1)$ passes from 1 to 0, equivalently as $\tau(1) = \sigma'(1) - 1$ passes from 0 to -1 , the curves $(r, \sigma(r))$ and $(r, \rho(r))$ pass monotonically from the lower boundary of Δ_σ to its upper boundary and, respectively, from the lower boundary of Δ_ρ to its upper boundary. The curves $(r, \tau(r))$ pass monotonically from the upper boundary of Δ_τ to the lower boundary.*

Proof That the solutions pass from one of the indicated extreme trajectories to the other follows from their continuity with respect to $\tau(1) = \sigma'(1) - 1$ - but at this point we could not be certain that trajectories remain within the indicated regions. Taking $\tau(1) = v$, the solution components $\tau(r)$, $\sigma(r)$ of (4.16) are differentiable with respect to v ; let the respective derivatives be $\tau_v(r)$, $\sigma_v(r)$. Then we have $\tau_v(1) = 1$, $\sigma_v(1) = 0$ as terminal conditions for the variational system derived from (4.16). Recalling that $\sigma'(r) = \tau + \sigma(r)/r$, that variational system is

$$-\frac{d\tau_v}{dr} = \frac{2\tau_v}{r} + \frac{2\tau\tau_v\beta(d)}{\sigma\gamma} + \frac{\tau^2(\beta(d))'d_v}{\sigma\gamma}$$

$$- \frac{\tau^2 \beta(d) \left(\sigma_v \gamma + \sigma \left((\beta(d))' d_v + 2\tilde{\lambda} (\tau + \sigma/r) (\tau_v + \sigma_v/r) \right) \right)}{\sigma^2 \gamma^2}, \quad (4.22)$$

$$\frac{d\sigma_v}{dr} = \left(\frac{\sigma_v}{r} + \tau_v \right), \quad (4.23)$$

where $\beta(d) = \alpha d^2 \phi''(d)$, $\tilde{\lambda} = \lambda + 2\mu$, $\gamma(r) = \beta(d) + \tilde{\lambda} (\tau + \sigma/r)^2$. This system, which is linear homogeneous in τ_v , σ_v , can be seen to take the form

$$- \frac{d\tau_v}{dr} = \left(\frac{2}{r} + \frac{\tau(r) \beta(d)}{\sigma(r) \gamma(r)} + \tilde{\psi}_1(r, \tau(r), \sigma(r)) \right) \tau_v + \tilde{\psi}_2(r, \tau(r), \sigma(r)) \sigma_v, \quad (4.24)$$

$$\frac{d\sigma_v}{dr} = \left(\frac{\sigma_v}{r} + \tau_v \right),$$

where $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are continuous functions of their arguments. More explicitly, using

$$\begin{aligned} \frac{d}{dv} (\beta(d)) &= (\beta(d))' d_\beta = (\beta(d))' \frac{d}{dv} \left(\frac{\sigma \sigma'}{r} \right) = (\beta(d))' \left(\frac{\sigma' \sigma_v + \sigma \sigma'_v}{r} \right) \\ (\beta(d))' \left(\frac{\sigma' \sigma_v + \sigma \left(\tau_v + \frac{\sigma_v}{r} \right)}{r} \right) &= (\beta(d))' \left(\left(\frac{\sigma'}{r} + \frac{\sigma}{r^2} \right) \sigma_v + \frac{\sigma}{r} \tau_v \right), \end{aligned}$$

we have

$$\begin{aligned} \tilde{\psi}_1(r, \tau(r), \sigma(r)) &= \frac{\tau^2 \left(\frac{\sigma^2}{r} \tilde{\lambda} \beta'(d) - 2\tilde{\lambda} \sigma' \beta(d) \right)}{\left(\sigma^2 \left(\beta(d) + \tilde{\lambda} (\sigma')^2 \right) \right)^2}, \\ \tilde{\psi}_2(r, \tau(r), \sigma(r)) &= \frac{\tau^2 \left(\sigma \beta'(d) \tilde{\lambda} (\sigma')^2 - \beta(d)^2 - \beta(d) \tilde{\lambda} (\sigma')^2 - \frac{2}{r} \sigma \sigma' \tilde{\lambda} \beta(d) \right)}{\left(\sigma^2 \left(\beta(d) + \tilde{\lambda} (\sigma')^2 \right) \right)^2}. \end{aligned} \quad (4.25)$$

The positivity of $d^2 \phi''(d)$ and non-positivity of $(d^2 \phi''(d))'$ of Assumption 4.1 together with the earlier demonstrated positivity of σ and σ' show that both quantities in (4.25) are non-positive. We have $\tau_v(1) = -1$, $\sigma_v(1) = 0$. Now we claim that $\sigma_v(r) > 0$, $r \in [r_0, 1)$. If not, since $\sigma_v(1) = 0$ and $\tau_v(1) = -1$, the equations (4.24) imply that $\sigma_v(r) > 0$ and $\tau_v < 0$ in some interval $(\hat{r}, 1]$, $r_0 < \hat{r} < 1$, of maximal positive length. Thus either $\sigma_v(\hat{r}) = 0$ or $\tau_v(\hat{r}) = 0$. The second equation of (4.24) together with the negativity of τ_v on $(\hat{r}, 1]$ shows the first alternative to be impossible because it implies

$$\sigma_v(\hat{r}) = -\hat{r} \int_{\hat{r}}^1 \frac{\tau_v(s)}{s} ds > 0.$$

On the other hand the negativity of $\tilde{\psi}_2$ in the first equation of (4.24) and the positivity of σ_v on $(\hat{r}, 1)$ together with $\tau_v(1) = -1$ shows that $\tau_v(\hat{r}) < 0$. Thus $\hat{r} > r_0$ leads to a contradiction and we conclude, for $0 < v < 1$, that

$$\sigma_v(r) > 0, \quad r_0 \leq r < 1, \quad \tau_v(r) < 0, \quad r_0 \leq r \leq 1.$$

From this the claim of the theorem follows for σ_v and τ_v . The result for $\rho(r) = \sigma(r) - r$ follows from the result for $\sigma(r)$.

Our final task is to examine the manner in which the pressure, p , applied at $r = r_0$, is related to $\tau(1) = v$. From (4.8) we have

$$p = - \left((\lambda + 2\mu) r_0 \rho'(r_0) + \lambda \rho(r_0) + \alpha (r_0 + \rho(r_0)) - \frac{\alpha r_0}{1 + \rho'(r_0)} \right). \quad (4.26)$$

The right hand side is a continuous function of $\sigma'(1)$ in the range $0 < \sigma'(1) \leq 1$, assuming the value 2α for $\sigma'(1) = 1$ and tending to ∞ as $\sigma'(1)$ tends to 0, corresponding to $\rho'(r)$ tending to -1 . Since $v = \tau(1) = \sigma'(1) - \sigma(1)/1$, $\sigma'(1)$ goes from 0 to 1 as v goes from -1 to 0. As $\sigma'(r) = 1 + \rho'(r)$ approaches 0 corresponding to $\sigma'(1) = 0$, $\tau(1) = -1$, it is clear from (4.26) that $p \rightarrow +\infty$. On the other hand for $\sigma(r) \equiv r$, $\rho(r) \equiv 0$, corresponding to $\sigma'(1) = 1$, $\tau(1) = 0$, we obtain $p = 0$.

Theorem 4.4 *If α is sufficiently small relative to λ and μ , every $p \in [0, \infty)$ corresponds to a unique value of $\sigma'(1)$ in the interval $(0, 1]$ (equivalently, a unique value of $\rho'(1) \in (-1, 0]$), each non-negative value of p being achieved just once. This further implies that $p = p(\rho(r_0))$ is a monotone increasing function of $\rho(r_0)$.*

Proof It is difficult to prove this result directly from (4.26), so we use an indirect argument. Suppose, for the moment, we could show that the integrand of (4.2) is strictly convex, as a function of $\rho(r)$, $\rho'(r)$ for each $r \in [r_0, 1]$. Then each trajectory $\rho(r)$, $r_0 \leq r \leq 1$, whatever the associated value of p , provides a strong local (at least) minimum for (4.2). This follows, cf. [14], e.g., from (easy) verification of the Weierstrass-Erdmann condition together with the fact that Theorem 4.2 shows the extrema $\rho(r)$ to be embedded in a field.

If there were a value of p corresponding, via (4.26), to two distinct values of $\rho'(1)$, thus to two distinct trajectories $\rho(r)$, $\hat{\rho}(r)$, each of these trajectories would afford a strong local minimum for (4.2). But it is a straightforward exercise, using $\rho_\epsilon(r) = (1 - \epsilon)\rho(r) + \epsilon\hat{\rho}(r)$, $0 < \epsilon < 1$, to see that strict convexity of the integrand of (4.2) with respect to $\rho(r)$, $\rho'(r)$ rules out at least one of these trajectories from being a strong local minimum. In this case, therefore, the relationship between $\rho'(1)$ and p must be one-to-one, hence p is a strictly decreasing function of $\rho'(1) \in (-1, 0]$, equivalently of $\sigma'(1) \in (0, 1]$.

From the monotonicity result for $\sigma_v(r)$ with respect to v obtained in Theorem 4.2 together with $\sigma(r_0) = r_0 + \rho(r_0)$, we conclude that $p = p(\rho(r_0))$ is a monotone increasing function of $\rho(r_0)$. Since the sum of the first two terms in the integrand of (4.2), i.e., the quadratic terms, is readily seen to be strictly convex for positive λ and μ , and since, for fixed r ,

$$-\log(\phi(r)) = -\log\left(1 + \frac{\rho(r)}{r}\right) - \log(1 + \rho'(r))$$

is a strictly convex function of $\rho(r)$, $\rho'(r)$ the integrand of (4.2) is strictly convex if $\alpha > 0$ is sufficiently small relative to λ and μ . This completes the proof.

Remark For given values of λ and μ the “sufficiently small” value of α will, in general, depend on r_0 . We will see that this can be a significant limiting factor in implementation of the method to the problem of §4.

5 Comments on Implementation

It would be desirable to have a convexity result independent of any assumptions on the relationship between λ , μ and α . From the discussion above it is clear that we have strict convexity of the integrand of (4.2) for all λ , μ and α just in case $\Phi(r) = \Phi(r, \rho(r), \rho'(r))$ is convex as a function of $\rho(r)$, $\rho'(r)$. Computing the Hessian matrix of

$$\Phi(r, y, z) = (1 + y/r)(1 + z) - \log\left(1 + \frac{y}{r}\right) - \log(1 + z)$$

with respect to y, z we obtain

$$\begin{pmatrix} (r+y)^{-2} & r^{-1} \\ r^{-1} & (1+z)^{-2} \end{pmatrix}.$$

Since the entries are non-negative, non-negativity of this matrix corresponds to non-negativity of the determinant

$$\frac{1}{(r+y)^2(1+z)^2} - \frac{1}{r^2} = \frac{1}{r^2} \left(\frac{1 - \left(1 + \frac{y}{r}\right)^2(1+z)^2}{\left(1 + \frac{y}{r}\right)^2(1+z)^2} \right),$$

true just in case $\left| \left(1 + \frac{y}{r}\right)(1+z) \right| \leq 1$. In our context this is the condition (cf. (4.3))

$$\phi(r) \equiv \left(1 + \frac{\rho(r)}{r}\right)(1 + \rho'(r)) < 1,$$

which, enforced for all $r \in [r_0, 1]$, is the requirement that the material should be everywhere under compression, or at least nowhere expanded. We have not been able to establish this property, without restrictions on the parameters as above, for the problem of §4. In numerical computations we have not experienced difficulties for r_0 fairly large, say $r_0 \geq .3$, provided the step size used in approximate integration of the system (4.16) is taken quite small - on the order of 10^{-3} to 10^{-4} . As r_0 is taken small, say $r_0 = .1$ or less, difficulties become substantially more pronounced. The reasons for this appear to be complex but include the presence of terms $1/r$ in the two differential equations of (4.16).

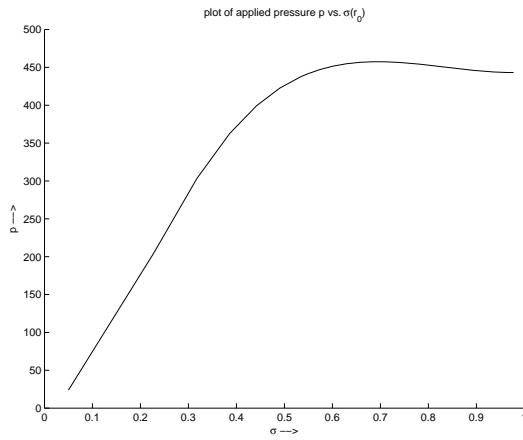


Figure 13

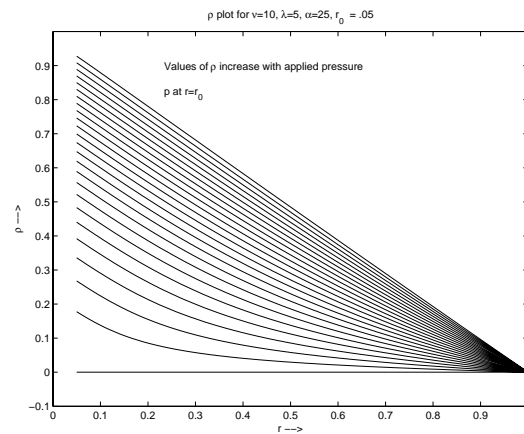


Figure 14

Typical results are illustrated graphically in Figures 13 - 16, all of which correspond to $\mu = 10$, $\lambda = 5$, $\alpha = 1$ and $r_0 = .2$. Figure 13 plots the applied pressure p versus the inner vacuole radius $\sigma(r_0) = r_0 + \rho(r_0)$. Figure 14 shows the corresponding field of trajectories $\rho(r)$ and Figure 15 the corresponding plot of the expansion/contraction ratios

$$c(r) = 1 + \rho'(r) + \frac{\rho(r)}{r} + \frac{\rho(r) \rho'(r)}{r}.$$

Finally Figure 16 shows the corresponding plots of the shear strain $\tau(r)$.

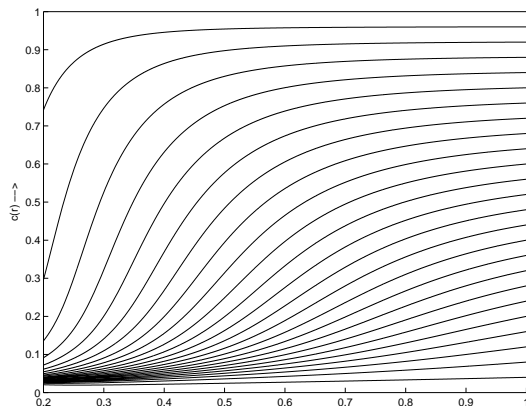


Figure 15

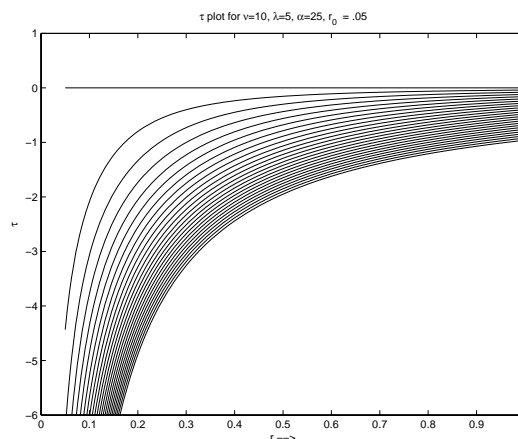


Figure 16

6 Conclusions

We have seen in §2, from computational experience, that the linear elastic model suffers from severe shortcomings, even in a qualitative sense, for relatively soft elastic bodies subjected to severe stresses or pressures. In §3 we have introduced certain *compression sensitive nonlinearities* to the elastic system by means of singular terms with a particular structure appended to the elastic potential energy in the Lagrangian. Computational experience with these added terms indicates much improved qualitative behavior of the modelled elastic systems. While we have not, at this time, presented a complete analysis of the behavior of elastic systems with these added terms, in §4 we have considered a rotationally invariant system in R^2 consisting of an annular elastic body subject to pressure applied in its central vacuole. In that context we have been able to demonstrate the global existence of solutions of the nonlinear equilibrium differential equations in the radial variable r and we have been able to establish uniqueness of solutions of the corresponding two point boundary value problem with fairly light additional assumptions.

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